Time Varying Electric and Magnetic fields

In time varying conditions, B can be represented in terms of vector potential

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{---} (1)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\therefore \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \quad \text{--} (2)$$

Since curl of a gradient of scalar function is zero, so from eq^n (2)

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\text{grad}\,\phi = -\nabla\phi$$

$$\text{or} \quad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{---} (3)$$

The two Homogeneous maxwell eq's are solved.

If A and ϕ are known, $\vec{B}$ and $\vec{E}$ can be obtained.

Eq^n (1) doesn't define A completely. If we add gradient of arbitrary scalar function in vector potential A;

$$\vec{A}' = \vec{A} + \nabla\psi \quad \text{--} (4)$$

The magnetic field remains invariant. What will happen to electric field $\vec{E}$? To ensure addition of $\nabla\psi$ in $\vec{A}$ does not make changes in electric field, it is necessary that the scalar potential also change to ϕ';

$$\phi' = \phi - \frac{\partial \psi}{\partial t} \quad \text{——} (5)$$

It can be verified by putting $\vec{A}'$ and ϕ' values from eq^n (4) and (5) in to eq^n (3).

Any physical law that can be expressed in terms of the electromagnetic potentials A and ϕ remains unaffected by the transformations of the type (4), (5).

These transformations are called "gauge transformations"

Clearly, eq^n's involving potentials must be gauge invariant.

In electrostatics we adopted the condition $\nabla \cdot A = 0$ which together with $B = \nabla \times A$ specified A.

In electromagnetism we have to make a different choice. In order to specify A we have to impose an additional condition on A in such a way that it does not change the physics. In other words, it must be consistent with the transformations (4) & (5) so that E and B remain unaffected.

We consider now, the two inhomogeneous Maxwell's eq^n's.

$$\nabla \cdot D = \nabla \cdot \varepsilon_0 E = \varepsilon_0 \nabla \cdot \left(-\nabla\phi - \frac{\partial A}{\partial t} \right) = \rho$$

$$\text{or} \quad -\nabla^2\phi - \frac{\partial}{\partial t}(\nabla \cdot A) = \rho/\varepsilon_0 \quad \text{---(6)}$$

and from eq^n

$$\nabla \times H = j + \frac{\partial D}{\partial t}$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = j$$

$$\nabla \times \frac{B}{\mu} - \varepsilon_0 \frac{\partial E}{\partial t} = j$$

$$\text{i.e.} \quad \frac{1}{\mu} \nabla \times (\nabla \times A) - \varepsilon_0 \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial A}{\partial t} \right) = j$$

i.e. $\nabla \times (\nabla \times A) - \mu\varepsilon_0 \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial A}{\partial t}\right) = \mu j$

i.e. $-\nabla^2 A + \nabla(\nabla\cdot A) + \mu\varepsilon_0 \frac{\partial}{\partial t}(\nabla\phi) + \mu\varepsilon_0 \frac{\partial^2 A}{\partial t^2} = \mu j$ ~(7)

We have used the identity $\nabla \times (\nabla \times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

~~i.e. $-\nabla^2 A + \nabla(\nabla\cdot A) +$~~

We choose A and ϕ such that

$$\nabla\cdot A = -\mu\varepsilon_0 \frac{\partial\phi}{\partial t} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t} \quad \text{--(8)}$$

With this substitution the two middle terms of eq^n (7) Cancel and the eq^n reduces to

$$\nabla^2 A - \mu\varepsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu j \quad \text{--(9)}$$

with the condition (8) eq^n (6) becomes

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla\cdot A) = -\nabla^2\phi + \mu\varepsilon_0 \frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon_0}$$

or $$\nabla^2\phi - \mu\varepsilon_0 \frac{\partial^2\phi}{\partial t^2} = -\rho/\varepsilon_0 \quad \text{--(10)}$$

The choice has yielded two independent eq^ns : one for A (eq^n (9)) and the other for ϕ (eq^n (10)). A is connected to vector j and ϕ with the scalar quantity ρ. further, both the eq^ns have the same form, i.e. both potentials satisfy the same equations.

The conditions thus, introduces complete symmetry between the scalar and vector potentials. for the steady-state, the time derivative vanish we have

$$\nabla^2 A = -\mu j \quad \text{and} \quad \nabla^2\phi = -\rho/\varepsilon_0$$

The condition, eqn (8) is known as Lorentz Guage Condition. The guage used in magnetostatics viz, $\nabla \cdot A = 0$ is called Coulomb Guage.

We have seen that the electric field E and magnetic field B are invariant under transformations (4) & (5). The potentials thus transformed will have to satisfy the Lorentz Condition. Hence, the guage function ψ which so far remained arbitrary will have to satisfy a certain condition.

Since the original and the transformed potentials have to satisfy Lorentz condition, we have

$$\nabla \cdot A + \mu \varepsilon_0 \frac{\partial \phi}{\partial t} = 0 \quad \text{--- (11)}$$

and

$$\nabla \cdot A' + \mu \varepsilon_0 \frac{\partial \phi'}{\partial t} = 0 \quad \text{--- (12)}$$

i.e. $$\nabla \cdot (A + \nabla \psi) + \mu \varepsilon_0 \frac{\partial}{\partial t}\left(\phi - \frac{\partial \psi}{\partial t}\right) = 0$$

i.e. $$\nabla \cdot A + \nabla^2 \psi + \mu \varepsilon_0 \frac{\partial \phi}{\partial t} - \mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} = 0$$

Hence $$\nabla^2 \psi - \mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \text{--- (13)}$$

Thus, the restricted guage transformation

$$A' \to A + \nabla \psi$$
$$\phi' \to \phi - \frac{\partial \psi}{\partial t} \quad \text{--- (14)}$$

where ψ satisfies the eqn

$$\nabla^2 \psi - \mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} = 0$$

preserve the Lorentz condition.